CHARACTER OF HEAT TRANSFER IN AN ACOUSTIC WAVE

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The use of hyperbolic-type heat-transfer equations in an acoustic wave is considered. Acoustic dispersion is calculated. Theoretical results are compared with experimental data on the coefficient of absorption for ultrasonic waves in castor oil.

1. Thermal-Conductivity Equation

Acoustic motions are a special case of hydrodynamic flows, so it follows that they may be calculated from the closed system of equations of hydrodynamics:

$$\rho \frac{d\mathbf{V}}{dt} = -\operatorname{grad} p + \mu\nabla^2\mathbf{V} + \frac{\mu}{3}\operatorname{grad}\operatorname{div}\mathbf{V}, \tag{1.1}$$

continuity:

$$\frac{d\rho}{dt} + \rho \operatorname{div}\mathbf{V} = 0, \tag{1.2}$$

state:

$$p = \varphi(\rho, T), \tag{1.3}$$

and energy:

$$\rho c_v \frac{dT}{dt} + p \operatorname{div}\mathbf{V} = -\operatorname{div}\mathbf{q}. \tag{1.4}$$

These equations consider the well-known Stokes hypothesis on the relationship of shear and volume viscosities.

With reference to acoustics, contemporary scientists dispute the validity of the equations of motion and state. In the equations of motion a volume viscosity totally unrelated to shear viscosity is introduced. Unfortunately, there are no means of obtaining the numerical value of this quantity other than acoustic experiments, which are the reasons for its existence in the first place. But this has disturbed no one, and the idea of a volume viscosity has been so firmly entrenched in contemporary molecular acoustics that one of the essential tenets of hydrodynamics has been neglected, whereby the deformation produced by isotropic compression of volume cannot be separated from deformation of its form. Thus, in analysis of any hydrodynamic process this fact should be stipulated, and if there is a desire to negate the Stokes postulate, the latter must be replaced by something. Meanwhile, no worker in contemporary molecular acoustics has acknowledged that a clear contradiction exists in this matter.

The disputed aspect of the equation of state is the introduction of an additional parameter, which in the equilibrium state is a function of specific volume and temperature. In the general case the parameter obeys some reaction equation.

However, another question may be put: In what form may the energy equation be used, without affecting the form of the equation of liquid motion and its equation of state? Indeed, such an alternative is also possible.

It is pertinent to recall that while refining the Stokes formula, in calculating acoustic dispersion Kirchhoff completed the energy equation in accordance with the Fourier hypothesis

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TABLE 1. Thermal Wave Velocity for Various Substances

Substance	Frequency ν, MHz	$\beta \cdot 10^8$ sec	G, m/sec	g_0, m/sec
Air	0,132—0,4	0,98—1,69	162—132	327
Carbon dioxide	0,304	1,93	20,1	259
Water	7—250	1,27	3,32	1490

$$\mathbf{q} = -\lambda \operatorname{grad} T. \tag{1.5}$$

In discussing this step he stressed that the quantity of heat is measured by that temperature increase which occurs in the absence of compression. In other words, the acoustic action does not affect thermal motions of the material system. However, the acoustic action may also be such that internal heat transfer between atomic groups of the molecules is disrupted. In this case the validity of Eq. (1.5) is doubtful. With respect to this we note that use of Eq. (1.5) implies an explicit dependence of thermal flux vector $\mathbf{q}$ on coordinates.

We propose that the acoustic action on internal heat transfer occurs in a manner such that the thermal flux vector becomes an explicit function of time; i. e., $\mathbf{q} = \mathbf{q}(x, y, z, t)$. We now differentiate the left and rights sides of Eq. (1.4) with respect to time:

$$c_v \frac{\partial}{\partial t}\left(\rho \frac{dT}{dt}\right) + \frac{\partial}{\partial t}(p \operatorname{div} \mathbf{V}) = -\frac{\partial}{\partial t} \operatorname{div} \mathbf{q}.$$

We multiply this equation by some quantity β and add the expression so obtained to Eq. (1.4). Then we have

$$\beta c_v \frac{\partial}{\partial t}\left(\rho \frac{dT}{dt}\right) + \beta \frac{\partial}{\partial t}(p \operatorname{div} \mathbf{V}) + \rho c_v \frac{dT}{dt} + p \operatorname{div} \mathbf{V} = -\operatorname{div} \mathbf{Q}. \tag{1.6}$$

Here the additional notation

$$\mathbf{Q}(x, y, z) = \mathbf{q} + \beta \frac{\partial \mathbf{q}}{\partial t} \tag{1.7}$$

is used. We assume that vector $\mathbf{Q}$ is a function of coordinates only and that the Fourier hypothesis may be applied to it; we then rewrite Eq. (1.6) as follows:

$$\beta c_v \frac{\partial}{\partial t}\left(\rho \frac{dT}{dt}\right) + \rho c_v \frac{dT}{dt} + \beta \frac{\partial}{\partial t}(p \operatorname{div} \mathbf{V}) + p \operatorname{div} \mathbf{V} = \lambda \nabla^2 T. \tag{1.8}$$

Here together with the normal coefficient of thermal conductivity λ there appears the new parameter β. If $\beta = 0$, then Eq. (1.8) transforms into a parabolic-type equation.

Equation (1.8) must be used when solutions of the classical equation of heat transfer contradict experiment. In this case, by using solutions of Eq. (1.8) and selecting a certain value for the parameter β, the contradiction between theory and experiment may be eliminated. Determining β in this manner and finding the constants by normal measurement techniques, we may determine the velocity of a thermal wave with the following formula:

$$G^2 = \frac{\lambda}{\rho c_v \beta} = \frac{a}{\beta}. \tag{1.9}$$

If we concretize the equation of state in the Clapeyron form

$$p = \rho R T,$$

then with consideration of the equation of continuity it can be shown that

$$p \operatorname{div} \mathbf{V} = R\rho \frac{dT}{dt} - \frac{dp}{dt}.$$

We then give to the heat-transfer equation the form

$$\frac{\beta c_p}{\rho c_v} \cdot \frac{\partial}{\partial t}\left(\rho \frac{dT}{dt}\right) - \frac{\beta}{\rho c_v} \cdot \frac{\partial}{\partial t} \cdot \frac{dp}{dt} + \frac{dT}{dt} + \frac{p}{\rho c_v} \operatorname{div} \mathbf{V} = a \nabla^2 T. \tag{1.10}$$

TABLE 2. Comparison of Theory and Experiment

ν, MHz	0°, C		10°, C	
	$\frac{\alpha}{\nu^2}\cdot 10^{13}$, sec²/cm		$\frac{\alpha}{\nu^2}\cdot 10^{13}$, sec²/cm	
	experiment	theory	experiment	theory
0,26	5,6	5,5	2,81	2,8
0,38	4,5	4,6	2,63	2,38
0.59	3,93	3,76	2,3	1,95
0,90	3,0	3,1	1,73	1,64
2.10	2,3	2,2	1,22	1,21
3,30	1,66	1,7	1,00	1,01
6.5	1,21	1,4	0,795	0,81
8,78	1,15	1,2	0,74	0,73
12,04	0,98	1,06	0,608	0,66
20,02	0,87	0,89	0,625	0,58
30,22	0,76	0,78	0,569	0,53

2. Acoustic Dispersion Calculation

We will consider the case of one-dimensional acoustic motions. Then, neglecting acoustic wind, we rewrite the basic system of equations as follows:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{4}{3}\mu\frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}=0, \tag{2.1}$$

$$\gamma\beta\frac{\partial^2 T}{\partial t^2}-\frac{\beta}{\rho c_v}\cdot\frac{\partial^2 p}{\partial t^2}+\frac{\partial T}{\partial t}+\frac{p}{\rho c_v}\cdot\frac{\partial u}{\partial x}=a\frac{\partial^2 T}{\partial x^2},$$

$$p=\varphi(\rho,\ T).$$

We use Hadamard's algorithm for calculation of acoustic dispersion. Details of this algorithm with reference to acoustic problems are presented in [1].

With the aid of homologous and kinematic Hugoniot–Hadamard conditions we rewrite the system of equations as follows:

continuity equation:

$$\lambda_{1\rho}g=\rho\lambda_{1u},$$

equation of motion:

$$\lambda_{1p}=\rho\lambda_{1u}\left(g+\frac{4}{3}\cdot\frac{\mu}{\rho}\cdot\frac{\lambda_{2u}}{\lambda_{1u}}\right), \tag{2.2}$$

equation of state:

$$\frac{\lambda_{1p}}{\lambda_{1\rho}}=\frac{\partial p}{\partial\rho}+\frac{\lambda_{1T}}{\lambda_{1\rho}}\cdot\frac{\partial p}{\partial T},$$

energy equation:

$$\frac{pg}{\rho^2 c_v}\lambda_{1\rho}=\lambda_{1T}\left[g+\frac{\lambda_{2T}}{\lambda_{1T}}(a-\gamma\beta g^2)+\frac{\lambda_{1p}}{\lambda_{1T}}\cdot\frac{\beta g^2}{\rho c_v}\cdot\frac{\lambda_{2p}}{\lambda_{1p}}\right].$$

The basic quantity which interests us in acoustic phenomena is the front displacement velocity g.

To eliminate arbitrary discontinuity parameters from Eq. (2.2) it is necessary to add the condition of periodicity in time of velocity, temperature, and pressure. It is simple to show that this condition has the form

$$\frac{\lambda_{2u}}{\lambda_{1u}}=\frac{\lambda_{2T}}{\lambda_{1T}}=\frac{\lambda_{2p}}{\lambda_{1p}}=-\frac{i\omega}{g}.$$

TABLE 3. Numerical Values of Experimental Constants

Temperature, °C	0	10	20	30	40
$B \cdot 10^{11}$	26,6	12,7	8,15	5,49	3,03
$C \cdot 10^{13}$	0,31	0,3	0,185	0,085	0,08

After eliminating the discontinuity parameters from Eq. (2.2), we will have

$$g^2(1+i\omega\beta) = g_0^2 + \frac{4}{3}\cdot\frac{\mu}{\rho}\cdot\frac{a\omega^2}{g^2} - \frac{4}{3}\cdot\frac{\mu}{\rho}\omega^2\beta + i\omega\left[\frac{4}{3}\cdot\frac{\mu}{\rho} + a\left(1-\frac{g_0^2}{\gamma g^2}\right) + \frac{\gamma^2(\gamma-1)+1}{\gamma}\beta g_0^2\right]. \tag{2.3}$$

Here the notation

$$g_0^2 = \frac{\partial p}{\partial \rho} + \frac{p}{\rho^2 c_v}\cdot\frac{\partial p}{\partial T} = \gamma\frac{\partial p}{\partial \rho} \tag{2.4}$$

is used. In order to separate real and imaginary parts of the equation we take

$$g = g_1 + ig_2.$$

Considering that $g_2 \ll g_1$ and dropping second-order terms [1], we obtain instead of Eq. (2.3)

$$g_1^2 = g_0^2,$$

$$2g_1g_2 = \omega\left[\frac{4}{3}\cdot\frac{\mu}{\rho} + \frac{(\gamma-1)}{\gamma}a + \frac{(\gamma-1)(\gamma^2-1)}{\gamma}\beta g_0^2\right]. \tag{2.5}$$

The formula for acoustic wave amplitude may be written as

$$A = A_0\exp\left[i\omega\left(t - \frac{xg_1}{g_1^2+g_2^2}\right)\right]\exp\left(-\frac{\omega g_2 x}{g_1^2+g_2^2}\right).$$

It is now simple to find the coefficient of sound wave absorption:

$$\alpha = \frac{\omega g_2}{g_1^2+g_2^2} \approx \frac{\omega g_2}{g_1^2}$$

or, defining g_1 and g_2 from Eq. (2.5), we obtain

$$\alpha = \frac{\omega^2}{2g_0^3}\left[\frac{4}{3}\cdot\frac{\mu}{\rho} + \frac{\gamma-1}{\gamma}a + \frac{(\gamma-1)(\gamma^2-1)}{\gamma}\beta g_0^2\right]. \tag{2.6}$$

If $\beta = 0$, then Eq. (2.6) transforms to the Stokes—Kirchhoff formula, but it follows from Eq. (1.9) that in this case the thermal wave velocity is infinite; i. e., the parabolic heat-transfer equation describes the process of wave propagation at infinite velocity. However, as a rule, natural heat-propagation processes possess inertia. It is thus possible that numerical values of α obtained from the Stokes—Kirchhoff formula will not coincide with experimental data even at frequencies where acoustic dispersion is absent. If we transform from angular frequency ω to cyclical frequency ν, then with consideration of Eq. (1.9) we may rewrite Eq. (2.6) as

$$\frac{\alpha}{\nu^2} = \left(\frac{\alpha}{\nu^2}\right)_0 + \frac{2\pi^2(\gamma-1)(\gamma^2-1)}{\gamma g_0}\cdot\frac{a}{G^2}. \tag{2.7}$$

Here additional notation has been introduced for the Stokes—Kirchhoff formula

$$\left(\frac{\alpha}{\nu^2}\right)_0 = \frac{2\pi^2}{g_0^3}\left(\frac{4}{3}\frac{\mu}{\rho} + \frac{(\gamma-1)}{\gamma}a\right). \tag{2.8}$$

Knowing the difference between experimental values of α/ν^2 and those calculated by Eq. (2.8), we can determine the thermal wave velocity G. Results of such calculations are presented in Table 1 for several gases at 0°C and water at 20°C. Experimental values of α/ν^2 were taken from [2] and values of the thermal diffusivity and heat capacity coefficients, from thermophysical measurements.

In analogy to the speed of sound, the speed of a thermal wave may depend on frequency; i. e., thermal dispersion can exist. Calculation of such a dependence is difficult at present, but it may be guessed by analysis of experimental data. We assume that

$$\frac{1}{G^2} = b_1 + \frac{b_2}{\sqrt{\nu}}. \tag{2.9}$$

Then we give to Eq. (2.7) the form

$$\frac{\alpha}{\nu^2} = \frac{B}{\sqrt{\nu}} + C, \tag{2.10}$$

where

$$B = \frac{2\pi^2(\gamma-1)(\gamma^2-1)ab_2}{\gamma g_0}; \quad C = \left(\frac{\alpha}{\nu^2}\right)_0 + b_1.$$

Equation (2.10) provides a good description of the experimental results of [3] on the absorption coefficient of ultrasonic waves in castor oil over a wide frequency range. The character of the agreement between theory and experiment is demonstrated by Table 2. Table 3 shows values of the constants B and C, determined by application of Eq. (2.10) to experimental results.

It is interesting that in [4] the following frequency dependence of the ultrasound absorption coefficient was found in an aqueous solution of polyethylene oxide:

$$\frac{\alpha}{\nu^2} \sim \frac{1}{\sqrt{\nu}},$$

which does not contradict Eq. (2.10). In this case the constants B and C will be functions of concentration and concentration waves should be considered.

NOTATION

μ, viscosity; λ, thermal conductivity; a, thermal diffusivity; c_v, heat capacity at constant volume; c_p, heat capacity at constant pressure; γ, heat capacity ratio; R, universal gas constant; λ_{1u}, λ_{1p}, $\lambda_{1\rho}$, λ_{1T}, first-order discontinuity parameters; λ_{2u}, λ_{2p}, $\lambda_{2\rho}$, λ_{2T}, second-order discontinuity parameters; g_0, Laplacian value of velocity of sound.

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